the time Eq. (1f) leads to an approximation with poor convergence characteristics. Hence it is advantageous to replace t by

$$t = t^* + \Delta t \qquad \text{with} \qquad t^* = \int_0^s R^{(0)}\, d\sigma = (1 - R^{(0)-1/2})/\varepsilon \tag{15}$$

For Δt an expansion solution may be found

$$\Delta t = -\varepsilon[R^{(0)1/2}(A\cos\tau - B\sin\tau) - \tfrac{9}{7}\times(1-R^{(0)}) + 2] \tag{16}$$

The two-variable expansion of the ϕ-equation (1e) gives a first-order approximation

$$\phi = \tau + \varepsilon R^{(0)-1}[(A\cos\tau - B\sin\tau) + \tfrac{9}{4}\times(R^{(0)2} - 1) + 2] \tag{17}$$

Both results Eqs. (16) and (17) include the assumption that the second-order expansion term of the energy, $H^{(2)}$, and that part of $v^{(2)}$ which depends on ϑ only, $V^{(2)}$, be zero, i.e. $H^{(2)} = V^{(2)} = 0$.

Discussion

The asymptotic solution shows the basic characteristics of the spiral trajectory. The zero-order solution $v^{(0)} = R^{(0)1/2}$ is equivalent to the relation $rv^2 = 1$, which usually is regarded as the standard circular asymptotic solution of spiral type trajectories.[6] Furthermore, the zero-order radius approximation is equivalent to the well-known spiral distance approximation $r^{(0)}(t) = (1-\varepsilon t)^{-2}$.

The derived solution has a singularity at $\vartheta = \frac{1}{3}$. This value corresponds to the escape condition $h = 0$ and leads to $r(\vartheta = \frac{1}{3}) \to \infty$. This is the well-known behavior of the Kepler solution of the equations of motion without thrust. Consequently, the solution is valid only for energy levels less than zero.

Near escape thrust acceleration and gravity force are of the same order of magnitude, and the solution ceases to possess two different time scales. This change in the physical characteristics violates the basic assumption for the procedure which implies the dominance of the gravity force. In addition the expansion solution assumes that the radial velocity component u is small compared to v. This condition does not hold near escape, too.

Examples

To check the accuracy, the analytic solution is compared with the results of Jacobson and Powers[3] and with numerical calculations. 1) Jacobson and Powers used the specifications $h_f = -0.2904134$ with $\varepsilon = 1.189409 \times 10^{-3}$. These lead to a scaled minimum time $t_f = 200$. The number of revolutions of the optimal trajectory is slightly more than 22. The coincidence of these results with the two-variable expansion solution lies between 3 and 6 significant digits for the different state variables and multipliers over the whole region $0 \leq t \leq t_f$. Figures 1 and 2 show the control angle and the radial distance histories as functions of the nondimensional time. One sees immediately

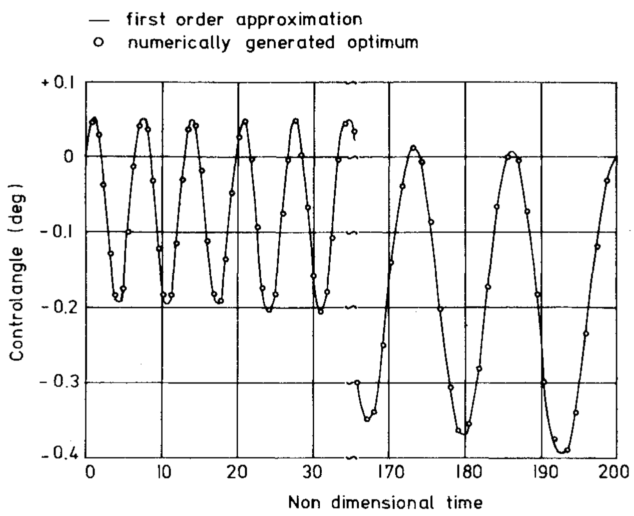

Fig. 1 Time history of the control angle (measured against the path tangent).

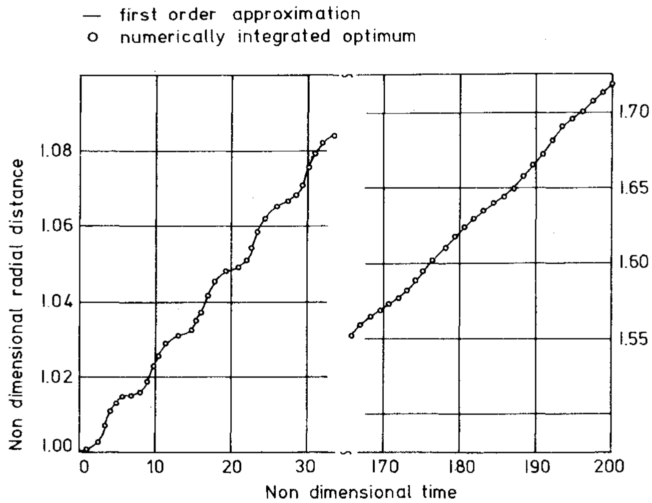

Fig. 2 Time history of the radial distance.

how well the approximation matches the changing amplitude and period of the trajectory and control angle oscillations.

2) The numerical procedure was calculated with $h_f = -0.195$ and $\varepsilon = 1.1242371 \times 10^{-4}$. This results in a scaled $t_f = 1\ 885.5$ and more than 300 revolutions. The analytical approximation and the numerically integrated trajectory coincide within at least 3 significant digits over the whole region. A numerical optimization program was started with Lagrange multipliers at $t_i = 0$ which resulted from the approximate solution. This extremely sensitive optimization process converged within few iterations. Time and centriangle approximations are accurate within 5 digits for both examples.

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Thermal Buckling of Orthotropic Cylindrical Shells

S. K. RADHAMOHAN* AND J. VENKATARAMANA*
Vikram Sarabhai Space Centre, Trivandrum, India

Introduction

THERMAL buckling plays an important role in the design of thin walled structures subjected to thermal environment. The existing studies on thermal buckling are confined to homogeneous, isotropic and ring stiffened shells.[1–7] Dasgupta and

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* Engineer, Structural Engineering Division.

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Wang[8] have evaluated the critical temperatures for short composite cylinders using an energy approach. The effect of prebuckling rotations, which influences the buckling behavior of short shells,[4,5] has not been included in Ref. 8. The purpose of this Note is to make a complete study of thermal buckling of shells, made up of composite materials like fibreglass reinforced plastics. The following aspects, which affect the buckling of a cylindrical shell subjected to a uniform temperature rise throughout and free from mechanical loads are studied: 1) effect of L/R ratio; 2) effect of prebuckling rotations; 3) effect of constraining the prebuckling meridional displacement; and 4) effect of different buckling boundary conditions.

Theory and Method of Solution

The governing differential equations of the prebuckling and buckling states are derived from Sanders' nonlinear thin shell theory.[9] To account for the effects of prebuckling rotations in short shells a nonlinear prebuckling analysis is carried out. The stress strain relations for an orthotropic material, including thermal effects are used in the present analysis.

The nonlinear differential equations of the prebuckled state are solved by a "Parametric Differentiation Technique," recently used in the investigation of shell buckling.[9,10] The method of solution is exactly similar to that described in Ref. 10 except that the uniform temperature is chosen as the parameter. The linear homogeneous differential equations of the buckled state are solved by carrying out four homogeneous integrations. The buckling criteria is the vanishing of the determinant formed by the coefficients of homogeneous integrations corresponding to the given boundary conditions.

Numerical Results and Discussion

In this Note, numerical results are obtained for materially orthotropic, clamped cylindrical shells. Though the material properties are strictly functions of temperature, as a first approximation, the assumed properties are representative in the temperature range under consideration. The properties chosen are: $E_1 = 1.91 \times 10^6$ lbs/in.^{2}; $E_2 = 5.02 \times 10^6$ lbs/in.^{2}; $\nu_1 = 0.25$; $G = 0.75 \times 10^6$ lbs/in.^{2}. The value of ν_2 can be determined from the reciprocal relation $E_1\nu_1 = E_2\nu_2$. Boundary conditions: The conditions chosen for the prebuckling state are:

$$w_o = 0;\ \phi_{so} = 0;\ N_{so} = 0 \quad \text{or} \quad u_o = 0 \quad \text{at ends}$$
$$u_o = 0;\ \phi_{so} = 0;\ Q_{so} = 0 \quad \text{at center of the shell.}$$

Both symmetric and antisymmetric modes of buckling are studied. To study the effect of different boundary conditions, the following clamped conditions are considered:

$$\left.\begin{aligned} w_n &= \phi_{sn} = u_n = v_n = 0 \quad &(c-1)\\ w_n &= \phi_{sn} = N_{sn} = v_n = 0 \quad &(c-2)\\ w_n &= \phi_{sn} = u_n = N_{s\theta n} = 0 \quad &(c-3)\\ w_n &= \phi_{sn} = N_{sn} = N_{s\theta n} = 0 \quad &(c-4)\end{aligned}\right\} \text{at the ends.}$$

where u, v, w are axial, circumferential and normal displacements; ϕ_s = meridional rotation; N_s, $N_{s\theta}$, Q_s are axial, circumferential and normal stress resultants; n = number of circumferential waves; subscripts o and n correspond to the prebuckling and the buckling states, respectively. In all the numerical examples discussed, the ratio of radius to thickness of shell is taken as 300.0.

Effect of curvature on critical temperature

Buckling parameters $\alpha_2 T_{cr}$ (where α_2 is the coefficient of thermal expansion in the circumferential direction; T_{cr} is the critical temperature) for various geometric parameters Z, are shown in Fig. 1. These results correspond to c-1 condition with axial displacement u_o released. Since u_o is allowed at supports, ϕ_{so}, and the circumferential stress resultant $N_{\theta o}$ are independent of α_1. The curve corresponding to symmetric mode resembles well that given by Chang and Card[5] for ring-stiffened shells. It is worth mentioning that the asymmetric mode predicts lower buckling values in the range $3.7 \leq Z \leq 26.6$. Earlier studies on thermal buckling of homogeneous and ring stiffened shells have

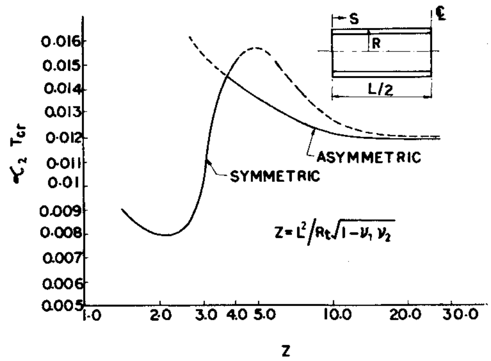

Fig. 1 Variation of critical temperature parameter with curvature.

not considered this aspect. For higher values of Z, the critical thermal load approaches a constant value 0.01196. The compressive hoop stress which causes buckling is found to be well within the elastic limit for all geometries chosen.

Effect of prebuckling rotations on buckling behavior

To study the effect of prebuckling rotation on composite shells, two geometries with $Z = 2.96$ and 26.6 are considered. The results are tabulated in Table 1. As in the case of homogeneous shells,[4,5] the inclusion of rotations yields higher buckling loads. This effect is more predominant for short shells. For the shell with $Z = 2.96$, the stresses just exceed the proportional limit, when prebuckling rotations are included. The shell deforms with less number of circumferential waves if the prebuckling rotations are not considered in the analysis.

Table 1 Effect of prebuckling rotations on $\alpha_2 T_{cr}(u_o$ constrained)

	With rotation		Without rotation	
Z	$\alpha_2 T_{cr} \times 10^4$	n	$\alpha_2 T_{cr} \times 10^4$	n
2.96	165.0	64	66.4	33
26.60	137.9	49	122.0	39

Effect of constraining u_o

This effect is studied for a shell with $Z = 26.6$ and for $\alpha_1/\alpha_2 = 0.3$, 1.0 and 1.5. In general, a compressive axial stress is induced due to the axial constraint. However it is observed that for $\alpha_1/\alpha_2 = 0.3$, the axial stress and meridional rotations (near the center of the shell) based on nonlinear analysis change their sign near the critical temperature. Figure 2 shows the nonlinear variation of meridional rotation (at $S = 0.4L$) with temperature. This observation necessitates the need to carry out a buckling analysis using the linear prebuckling state. Such an analysis predicts $\alpha_2 T_{cr}$ as 85.9, a much lower value than nonlinear

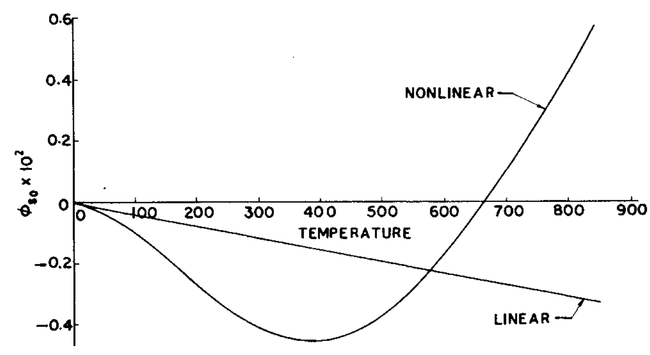

Fig. 2 Variation of prebuckling rotation at $S/L = 0.4$ with temperature.

Table 2 Effect of α_1/α_2 on $\alpha_2 T_{cr}(Z = 26.6)$

α_1/α_2	u_o restrained		u_o unrestrained	
	$\alpha_2 T_{cr}$	n	$\alpha_2 T_{cr}$	n
0.3	110.0	44	119.6	47
	85.9[a]	36		
1.0	138.0	49	119.6	47
1.5	151.0	50	119.6	47

[a] Linear prebuckling analysis.

analysis. Also the shell deforms into smaller number of circumferential waves ($n = 36$). It can be seen from Table 2 that the parameter $\alpha_2 T_{cr}$ increases in magnitude for increasing values of α_1/α_2 ratio.

Effect of boundary conditions on $\alpha_2 T_{cr}$

The critical parameter $\alpha_2 T_{cr}$ for a shell with $Z = 1.89$, for all the boundary conditions c-1 to c-4 are given in Table 3. It can be seen that the critical temperature of a cylinder, constrained axially is unaffected whether the circumferential edge is restrained or free. If the axial displacement is released, the buckling parameter is lower when the displacement v_n is unrestrained than when restrained. However the difference is not considerable. Irrespective of the boundary conditions used in the circumferential direction, the number of circumferential waves for axially restrained cylinder is more than the unrestrained case.

Table 3 Effect of different buckling conditions on $\alpha_2 T_{cr}(Z = 1.89)$

Boundary conditions	$\alpha_2 T_{cr}$	n
c-1	80.87	54
c-2	80.20	50
c-3	80.70	54
c-4	78.10	49

Conclusions

The results for clamped cylinders indicate that for lower values of geometric parameter ($Z \leqq 3.7$) the symmetric mode of buckling is predominant while for $Z \leqq 3.7$, the asymmetric mode governs the behavior. Inclusion of prebuckling rotations in the analysis estimates higher buckling temperature. The critical temperature is higher for increasing values of the α_1/α_2 ratio. The buckling boundary conditions do not significantly alter the buckling behavior.

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Prevention of Delamination of Composite Laminates

N. J. PAGANO*
Air Force Materials Laboratory, Dayton, Ohio
AND
L. M. LACKMAN†
Rockwell International, Los Angeles, Calif.

As described by Dickerson and Lackman,[1] the longerons of the B-1 air vehicle consist of a basic metal element with an adhesively bonded boron/epoxy cross-ply laminate. Concern with regard to free edge delamination caused by interlaminar normal stress σ_z[2-4] has led to a design in which the 90° layers have been "softened" in the vicinity of the free edges by means of a mechanical serrating process as shown in Fig. 1. In this Note, we shall present some experimental observations with respect to the effectiveness of this procedure for large diameter boron/epoxy laminates under static and fatigue (tension) loadings. Specifically, we shall consider the extreme case of a laminate that is highly prone toward delamination and observe the influence of serration on its delamination threshold, ultimate strength, and fatigue life.

It should be noted that one approach to prohibit delamination has already been proposed in the literature.[2] This involves the prescription of a detailed stacking sequence such that inter-

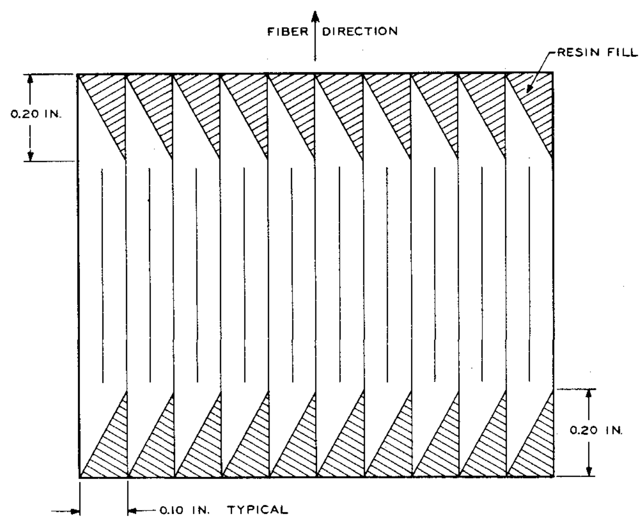

Fig. 1 Serration pattern.

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* Materials Research Engineer, Nonmetallic Materials Division.

† Manager, Composite Programs, Los Angeles Aircraft Division.